

PERTURBATION ANALYSIS OF $A_{T,S}^{(2)}$ ON BANACH SPACES

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Abstract. In this paper, the perturbation problems of $A_{T,S}^{(2)}$ are considered. By virtue of the gap between subspaces, we derive the conditions that make the perturbation of $A_{T,S}^{(2)}$ is stable when T, S and A have suitable perturbations. At the same time, the explicit formulas for perturbation of $A_{T,S}^{(2)}$ and new results on perturbation bounds are obtained.

Key words. gap, subspaces, Banach space, group inverse

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1. Introduction. In recent years, there are many fruitful results concerning the quantitative analysis of the perturbation of the Moore–Penrose inverses on Hilbert spaces and Drazin inverses on Banach spaces. For example, G. Chen, M. Wei and Y. Xue gave an estimation of the perturbation bounds of the Moore–Penrose inverse on Hilbert spaces under stable perturbation of operators, which is a generalization of the rank–preserving perturbation of matrices in [1, 15]; Meanwhile, many perturbation analysis results of the Drazin inverse on Banach spaces have been obtained in [2, 3, 4] and [8] respectively by means of the gap–function. Recently, G. Chen and Y. Xue gave some estimations of the perturbations of the Drazin inverse on a Banach space and a Banach algebra in [13] and [16] respectively under stable perturbations.

Let X, Y be Banach spaces and let $B(X, Y)$ denote the set of bounded linear operators from X to Y . For an operator $A \in B(X, Y)$, let $R(A)$ and $N(A)$ denote the range and kernel of A , respectively. Let T be a closed subspace of X and S be a closed subspace of Y . Recall that $A_{T,S}^{(2)}$ is the unique operator G satisfying

$$(1.1) \quad GAG = G, \quad R(G) = T, \quad N(G) = S.$$

It is known that (1.1) is equivalent to the following condition:

$$(1.2) \quad N(A) \cap T = \{0\}, \quad AT \dot{+} S = Y$$

(cf. [6, 7]). It is well-known that the commonly five kinds of generalized inverses: the Moore–Penrose inverse A^+ , the weighted Moore–Penrose inverse A_{MN}^+ , the Drazin

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inverse A^D , the group inverse A^g and the Bott–Duffin inverse $A_{(L)}^{(-1)}$ can be reduced to an $A_{T,S}^{(2)}$ for certain choices of T and S (cf. [5, 6, 7]).

The perturbation analysis of $A_{T,S}^{(2)}$ have been studied by several authors (see [11, 12], [17, 18]) when X and Y are of finite-dimensional. A lot of results about the error bounds have been obtained. But when X and Y are of infinite-dimensional, there is little known about the perturbation of $A_{T,S}^{(2)}$ if T , S and A have small perturbations respectively. In this paper, using the gap-function $\hat{\delta}(\cdot, \cdot)$ of two closed subspaces, we give the the upper bounds of $\|\bar{A}_{T',S'}^{(2)}\|$ and $\|\bar{A}_{T',S'}^{(2)} - A_{T,S}^{(2)}\|$ respectively. The main result is the following:

Let $A, \bar{A} = A + E \in B(X, Y)$ and let $T \subset X$, $S \subset Y$ be closed subspaces such that $A_{T,S}^{(2)}$ exists. Let $T' \subset X$, $S' \subset Y$ be closed subspaces with $\hat{\delta}(T, T') < \frac{1}{(1+\kappa)^2}$ and $\hat{\delta}(S, S') < \frac{1}{3+\kappa}$. Suppose that $\|A_{T,S}^{(2)}\| \|E\| < \frac{2\kappa}{(1+\kappa)(4+\kappa)}$. Then $\bar{A}_{T',S'}^{(2)}$ exists and

$$\begin{aligned} \|\bar{A}_{T',S'}^{(2)}\| &\leq \frac{(1 + \hat{\delta}(S', S)) \|A_{T,S}^{(2)}\|}{1 - (1 + \kappa)\hat{\delta}(T, T') - \kappa\hat{\delta}(S', S) - (1 + \hat{\delta}(S', S)) \|A_{T,S}^{(2)}\| \|E\|}, \\ \frac{\|\bar{A}_{T',S'}^{(2)} - A_{T,S}^{(2)}\|}{\|A_{T,S}^{(2)}\|} &\leq \frac{(1 + \kappa)(\hat{\delta}(T, T') + \hat{\delta}(S', S) + (1 + \hat{\delta}(S', S)) \|A_{T,S}^{(2)}\| \|E\|)}{1 - (1 + \kappa)\hat{\delta}(T, T') - \kappa\hat{\delta}(S', S) - (1 + \hat{\delta}(S', S)) \|A_{T,S}^{(2)}\| \|E\|}, \end{aligned}$$

where $\kappa = \|A\| \|A_{T,S}^{(2)}\|$ is called the condition number of $A_{T,S}^{(2)}$. These results improve Theorem 4.4.5 of [14].

2. Preliminaries. Let Z be a complex Banach space. Let M, N be two closed subspaces in Z . Set

$$\delta(M, N) = \begin{cases} \sup\{\text{dist}(x, N) \mid x \in M, \|x\| = 1\}, & M \neq \{0\} \\ 0 & M = \{0\} \end{cases},$$

where $\text{dist}(x, N) = \inf\{\|x - y\| \mid y \in N\}$. The gap $\hat{\delta}(M, N)$ of M, N is given by $\hat{\delta}(M, N) = \max\{\delta(M, N), \delta(N, M)\}$. For convenience, we list some properties about $\delta(M, N)$ and $\hat{\delta}(M, N)$ which come from [9] as follows.

PROPOSITION 2.1. *Let M, N be closed subspaces in a Banach space Z . Then*

- (1) $\delta(M, N) = 0$ if and only if $M \subset N$.
- (2) $\hat{\delta}(M, N) = 0$ if and only if $M = N$.
- (3) $\hat{\delta}(M, N) = \hat{\delta}(N, M)$.
- (4) $0 \leq \delta(M, N) \leq 1, 0 \leq \hat{\delta}(M, N) \leq 1$.

An operator $A \in B(Z, Z)$ is group invertible if there is $B \in B(Z, Z)$ such that

$$ABA = A, \quad BAB = B, \quad AB = BA.$$

The operator B is called the group inverse of A and is denoted by A^g . Clearly, $R(A^g) = R(A)$ and $N(A^g) = N(A)$.

LEMMA 2.2. *Let $A \in B(X, Y)$. Let $T \subset X$, $S \subset Y$ be closed subspaces such that $A_{T,S}^{(2)}$ exists. Let $G \in B(Y, X)$ be an operator with $R(G) = T$ and $N(G) = S$. Then*

- (1) $R(AG) = AT$, $N(AG) = S$ and $R(GA) = T$, $N(GA) \cap T = \{0\}$.
- (2) GA and AG are group invertible and $A_{T,S}^{(2)} = (GA)^g G = G(AG)^g$.

Proof. (1) Using $AT \dot{+} S = Y$ and $N(A) \cap T = \{0\}$, we can obtain the assertion.

(2) The assertion follows from [5, Lemma 3.1]. \square

LEMMA 2.3 ([10, Theorem 11, pp. 100]). *Let M be a complemented subspace of X . Let $P \in B(X, X)$ be an idempotent operator with $R(P) = M$. Let M' be a closed subspace of H satisfying $\hat{\delta}(M, M') < \frac{1}{1 + \|P\|}$. Then M' is complemented, that is, $H = R(I - P) \dot{+} M'$.*

Let $A \in B(X, Y)$. Let $T \subset X$ and $S \subset Y$ be closed subspaces such that $A_{T,S}^{(2)}$ exists. Put $\kappa = \|A\| \|A_{T,S}^{(2)}\|$. The symbol κ will be used throughout the paper.

LEMMA 2.4. *Let $A \in B(X, Y)$. Let $T \subset X$ and $S \subset Y$ be closed subspaces such that $A_{T,S}^{(2)}$ exists. Let T' be a closed subspace of X such that $\hat{\delta}(T, T') < \frac{1}{1 + \kappa}$. Then*

- (1) $\hat{\delta}(AT, AT') \leq \frac{\kappa \hat{\delta}(T, T')}{1 - (1 + \kappa) \hat{\delta}(T, T')}$.
- (2) $N(A) \cap T' = \{0\}$.

Proof. (1) First we show $\delta(AT, AT') \leq \|A\| \|A_{T,S}^{(2)}\| \delta(T, T') \leq \kappa \hat{\delta}(T, T')$.

Let $x \in T$. Then $x = A_{T,S}^{(2)} Ax$ and $\|x\| \leq \|A_{T,S}^{(2)}\| \|Ax\|$. For any $y \in T'$, we have $\|Ax - Ay\| \leq \|A\| \|x - y\|$. So

$$\begin{aligned} \text{dist}(Ax, AT') &= \inf_{y \in T'} \|Ax - Ay\| \leq \|A\| \inf_{y \in T'} \|x - y\| \\ &= \|A\| \text{dist}(x, T') \leq \|A\| \|x\| \delta(T, T') \\ &\leq \|A\| \|A_{T,S}^{(2)}\| \|Ax\| \delta(T, T'). \end{aligned}$$

This means that $\delta(AT, AT') \leq \kappa \delta(T, T') \leq \kappa \hat{\delta}(T, T')$.

Next we show $\delta(AT', AT) \leq \frac{\kappa \hat{\delta}(T, T')}{1 - (1 + \kappa) \hat{\delta}(T, T')}$ when $\hat{\delta}(T, T') < \frac{1}{1 + \kappa}$.

For each $x' \in T'$ and $x \in T$, we have

$$\begin{aligned}
\|Ax'\| &= \|A(x' - x + x)\| \geq \|Ax\| - \|A\|\|x' - x\| \\
&\geq \|A_{T,S}^{(2)}\|^{-1}\|x\| - \|A\|\|x' - x\| \\
&\geq \|A_{T,S}^{(2)}\|^{-1}\|x'\| - \|A_{T,S}^{(2)}\|^{-1}\|x' - x\| - \|A\|\|x' - x\| \\
&\geq \|A_{T,S}^{(2)}\|^{-1}\|x'\| - (\|A_{T,S}^{(2)}\|^{-1} + \|A\|)\|x' - x\|,
\end{aligned}$$

Thus,

$$(\|A_{T,S}^{(2)}\|^{-1} + \|A\|)\|x' - x\| \geq \|A_{T,S}^{(2)}\|^{-1}\|x'\| - \|Ax'\|$$

and consequently,

$$\|A_{T,S}^{(2)}\|^{-1}\|x'\| - \|Ax'\| \leq \|x'\|(\|A_{T,S}^{(2)}\|^{-1} + \|A\|)\delta(T', T),$$

that is,

$$(2.1) \quad \|A_{T,S}^{(2)}\|\|Ax'\| \geq [1 - (1 + \|A\|\|A_{T,S}^{(2)}\|)\delta(T', T)]\|x'\|.$$

Therefore,

$$\begin{aligned}
\text{dist}(Ax', AT) &\leq \|A\|\text{dist}(x', T) \leq \|A\|\|x'\|\delta(T', T) \\
&\leq \frac{\|A\|\|Ax'\|\|A_{T,S}^{(2)}\|\hat{\delta}(T, T')}{1 - (1 + \|A\|\|A_{T,S}^{(2)}\|)\hat{\delta}(T, T')},
\end{aligned}$$

$$\text{i.e., } \delta(AT', AT) \leq \frac{\kappa \hat{\delta}(T, T')}{1 - (1 + \kappa)\hat{\delta}(T, T')} \text{ when } \hat{\delta}(T, T') < \frac{1}{1 + \kappa}.$$

The final assertion follows from above arguments.

(2) From (2.1), we get that $N(A) \cap T' = \{0\}$. \square

3. Main results. .

LEMMA 3.1. *Let $A \in B(X, Y)$ and let $T \subset X$, $S \subset Y$ be closed subspaces such that $A_{T,S}^{(2)}$ exists. Let T' be closed subspace in X with $\hat{\delta}(T, T') < \frac{1}{(1 + \kappa)^2}$. Then $A_{T',S}^{(2)}$ exists and*

- (1) $A_{T',S}^{(2)} = A_{T,S}^{(2)} + (I - A_{T,S}^{(2)}A)F(I + (AG)^g AF)^{-1}(AG)^g$, where $G, H \in B(Y, X)$ are arbitrary operators such that

$$R(G) = T, \quad R(H) = T', \quad N(G) = N(H) = S \quad \text{and} \quad F = H - G.$$

$$(2) \|A_{T',S}^{(2)} - A_{T,S}^{(2)}\| \leq \frac{(1+\kappa)\hat{\delta}(T, T')}{1 - (1+\kappa)\hat{\delta}(T, T')} \|A_{T,S}^{(2)}\|.$$

$$(3) \|A_{T',S}^{(2)}\| \leq \frac{\|A_{T,S}^{(2)}\|}{1 - (1+\kappa)\hat{\delta}(T, T')}.$$

Proof. Put $P_{AT,S} = AA_{T,S}^{(2)}$. Then $P_{AT,S}$ is an idempotent operator onto AT along S . By Lemma 2.4 (1), we have

$$\hat{\delta}(AT, AT') \leq \frac{\kappa \hat{\delta}(T, T')}{1 - (1+\kappa)\hat{\delta}(T, T')} < \frac{1}{1+\kappa} \leq \frac{1}{1 + \|P_{AT,S}\|},$$

when $\hat{\delta}(T, T') < \frac{1}{(1+\kappa)^2}$. So AT' is complemented and $AT' \dot{+} S = Y$ by Lemma 2.3.

Consequently, $A_{T',S}^{(2)}$ exists by Lemma 2.4 (2).

Let $G, H \in B(Y, X)$ with $R(G) = T$, $N(G) = N(H) = S$ and $R(H) = T'$. Then by Lemma 2.2, we have

$$A_{T,S}^{(2)} = G(AG)^g = (GA)^g G, \quad A_{T',S}^{(2)} = H(AH)^g = (HA)^g H.$$

Put $F = H - G$. Then $S \subseteq N(F)$.

Now we show that $I + (AG)^g AF$ is invertible. Let $y \in N(I + (AG)^g AF)$. Then

$$y = -(AG)^g AFy \in R((AG)^g) = R(AG) = AT.$$

Hence

$$AA_{T,S}^{(2)}y = y = -(AG)^g AFy = AA_{T,S}^{(2)}y - (AG)^g AHy.$$

So $(AG)^g AHy = 0$. This indicates that

$$AHy \in R(AH) \cap N((AG)^g) = AT' \cap S = \{0\}.$$

From $AHy = 0$, we get that $y \in N(AH) \cap AT = S \cap AT = \{0\}$, i.e. $y = 0$. Therefore $I + (AG)^g AF$ is injective.

Note that $N((AG)^g) = S$ and $AT' \dot{+} S = Y$. So

$$AT = R(AG) = R((AG)^g) = (AG)^g AT' = R((AG)^g AH)$$

and consequently, for any $y \in Y = S \dot{+} AT$, there is $y_1 \in S$ and $y_2 \in R((AG)^g AH)$ such that $y = y_1 + y_2$. Choose $z \in Y$ such that $y_2 = (AG)^g AH z$. Write $z = z_1 + z_2$ where $z_1 \in AT$ and $z_2 \in S$. Since $N(H) = S$, $y_2 = (AG)^g AH z_1$. Set $\xi = y_1 + z_1$. Then

$$\begin{aligned} (I + (AG)^g AF)\xi &= (I - AA_{T,S}^{(2)} + (AG)^g AH)\xi \\ &= (I - AA_{T,S}^{(2)})\xi + (AG)^g AH\xi = y_1 + (AG)^g AH z_1 \\ &= y_1 + y_2 = y, \end{aligned}$$

that is, $I + (AG)^g AF$ is surjective. Therefore, $I + (AG)^g AF$ is invertible and $I + AF(AG)^g$ is invertible too.

Put

$$D = A_{T,S}^{(2)} + (I - A_{T,S}^{(2)}A)F(I + (AG)^g AF)^{-1}(AG)^g.$$

It is easy to verify that $DAD = D$ and $N(D) = S$. Since $(I + (AG)^g AF)^{-1}(AG)^g = (AG)^g(I + AF(AG)^g)^{-1}$ and

$$\begin{aligned} D &= A_{T,S}^{(2)} + (I - A_{T,S}^{(2)}A)F(I + (AG)^g AF)^{-1}(AG)^g \\ &= G(AG)^g + (I - G(AG)^g A)F(I + (AG)^g AF)^{-1}(AG)^g AG(AG)^g \\ &= (G + G(AG)^g AF + F - G(AG)^g AF)(I + (AG)^g AF)^{-1}(AG)^g \\ &= H(I + (AG)^g AF)^{-1}(AG)^g \\ &= H(AG)^g(I + AF(AG)^g)^{-1} \end{aligned}$$

by Lemma 2.2 (2), we have that

$$R(D) = R(H(AG)^g) = H(AT) = H(AT \dot{+} S) = R(H) = T'.$$

Thus $A_{T',S}^{(2)} = D$.

Put $W = A_{T',S}^{(2)} - A_{T,S}^{(2)}$. For any $\xi \in Y = AT' \dot{+} S$, there is $u \in AT'$ and $u' \in S$ such that $\xi = u + u'$. Choose $x \in Y$ such that $u = AA_{T',S}^{(2)}x$. Since $\text{dist}(A_{T',S}^{(2)}x, T) \leq \|A_{T',S}^{(2)}x\|\delta(T', T)$, for every $\epsilon > 0$, we can find $y \in Y$ such that

$$\|A_{T',S}^{(2)}x - A_{T,S}^{(2)}y\| < \|A_{T',S}^{(2)}x\|\delta(T', T) + \epsilon.$$

Set $v = AA_{T,S}^{(2)}y$. Then

$$\|u - v\| = \|AA_{T',S}^{(2)}x - AA_{T,S}^{(2)}y\| < \|A\|\|A_{T',S}^{(2)}x\|\delta(T', T) + \|A\|\epsilon.$$

Consequently,

$$\begin{aligned} \|W\xi\| &= \|Wu\| = \|A_{T',S}^{(2)}u - A_{T,S}^{(2)}u\| \\ &\leq \|A_{T',S}^{(2)}u - A_{T,S}^{(2)}v\| + \|A_{T,S}^{(2)}u - A_{T,S}^{(2)}v\| \\ &\leq \|A_{T',S}^{(2)}x - A_{T,S}^{(2)}y\| + \|A_{T,S}^{(2)}\|\|u - v\| \\ (3.1) \quad &\leq (1 + \kappa)\|A_{T',S}^{(2)}x\|\delta(T', T) + (1 + \kappa)\epsilon. \end{aligned}$$

Since

$$(3.2) \quad \|A_{T',S}^{(2)}\xi\| = \|A_{T',S}^{(2)}u\| = \|W\xi + A_{T,S}^{(2)}\xi\| \leq \|A_{T,S}^{(2)}\|\|\xi\| + \|W\xi\|,$$

it follows from (3.1) and (3.2) that

$$\|W\xi\| \leq (1 + \kappa)(\|A_{T,S}^{(2)}\|\|\xi\| + \|W\xi\|)\delta(T', T) + (1 + \kappa)\epsilon.$$

and hence $\|W\xi\| \leq \frac{(1 + \kappa)\delta(T', T)}{1 - (1 + \kappa)\delta(T', T)}\|A_{T,S}^{(2)}\|\|\xi\|$ by letting $\epsilon \rightarrow 0^+$. Therefore,

$$\|A_{T',S}^{(2)} - A_{T,S}^{(2)}\| \leq \frac{(1 + \kappa)\hat{\delta}(T', T)}{1 - (1 + \kappa)\hat{\delta}(T', T)}\|A_{T,S}^{(2)}\|.$$

Furthermore,

$$\begin{aligned} \|A_{T',S}^{(2)}\| &= \|W + A_{T,S}^{(2)}\| \leq \|W\| + \|A_{T,S}^{(2)}\| \\ &\leq \frac{(1 + \kappa)\hat{\delta}(T', T)}{1 - (1 + \kappa)\hat{\delta}(T', T)}\|A_{T,S}^{(2)}\| + \|A_{T,S}^{(2)}\| \\ &= \frac{\|A_{T,S}^{(2)}\|}{1 - (1 + \kappa)\hat{\delta}(T', T)}. \quad \square \end{aligned}$$

LEMMA 3.2. *Let $A \in B(X, Y)$ and let $T \subset X$, $S \subset Y$ be closed subspaces such that $A_{T,S}^{(2)}$ exists. Let S' be a closed subspace in Y such that $\hat{\delta}(S, S') < \frac{1}{2 + \kappa}$. Then $A_{T,S'}^{(2)}$ exists and*

- (1) $A_{T,S'}^{(2)} = A_{T,S}^{(2)} + A_{T,S}^{(2)}(I + (AG)^g AF)^{-1}(AG)^g AF(I - AA_{T,S}^{(2)})$, where $F = H - G$ and $G, H \in B(Y, X)$ are arbitrary with $R(G) = R(H) = T$, $N(G) = S$ and $N(H) = S'$.
- (2) $\|A_{T,S}^{(2)} - A_{T,S'}^{(2)}\| \leq \frac{(1 + \kappa)\hat{\delta}(S', S)}{1 - \kappa\hat{\delta}(S', S)}\|A_{T,S}^{(2)}\|.$
- (3) $\|A_{T,S'}^{(2)}\| \leq \frac{1 + \hat{\delta}(S', S)}{1 - \kappa\hat{\delta}(S', S)}\|A_{T,S}^{(2)}\|.$

Proof. Let $P_{S,AT} = I - AA_{T,S}^{(2)}$ be an idempotent operator from Y onto S along AT . Since $\|P_{S,AT}\| \leq 1 + \|A\|\|A_{T,S}^{(2)}\| = 1 + \kappa$, we have $\hat{\delta}(S, S') \leq \frac{1}{1 + \|P_{S,AT}\|}$. So $Y = AT \dot{+} S'$ by Lemma 2.3. Noting that $N(A) \cap T = \{0\}$, we get that $A_{T,S'}^{(2)}$ exists.

Using the facts:

$$AT \dot{+} S = Y = AT \dot{+} S', \quad N(A) \cap T = \{0\}$$

and the similar method appeared in the proof of Lemma 3.1, we can deduce that $I + (AG)^g AF$ is invertible and so is the operator $I + AF(AG)^g$.

Put $D = A_{T,S}^{(2)}(I + (AG)^g AF)^{-1}(AG)^g AH$. Then $R(D) \subset T$, $S' \subset N(D)$ and

$$\begin{aligned}
 & A_{T,S}^{(2)} + A_{T,S}^{(2)}(I + (AG)^g AF)^{-1}(AG)^g AF(I - AA_{T,S}^{(2)}) \\
 &= A_{T,S}^{(2)} + A_{T,S}^{(2)}(I + (AG)^g AF)^{-1}[I + (AG)^g AF - I](I - AA_{T,S}^{(2)}) \\
 &= A_{T,S}^{(2)}(I + (AG)^g AF)^{-1}[I + (AG)^g AF - (I - AA_{T,S}^{(2)})] \\
 (3.3) \quad &= A_{T,S}^{(2)}(I + (AG)^g AF)^{-1}(AG)^g AH.
 \end{aligned}$$

Clearly $DAD = D$ by (3.3). In order to obtain $A_{T,S}^{(2)} = D$, we need only to prove that $T \subset R(D)$ and $S' \supset N(D)$.

Since $AT \dot{+} S = Y$ and $N((AG)^g) = S$, $R(H) = T$, it follows that

$$R((AG)^g) = (AG)^g AT = R((AG)^g AH)$$

and hence

$$\begin{aligned}
 R(D) &= A_{T,S}^{(2)}(I + (AG)^g AF)^{-1}R((AG)^g) = R(A_{T,S}^{(2)}(AG)^g(I + AF(AG)^g)^{-1}) \\
 &= A_{T,S}^{(2)}R((AG)^g) = A_{T,S}^{(2)}AT = A_{T,S}^{(2)}(AT \dot{+} S) = R(A_{T,S}^{(2)}) = T.
 \end{aligned}$$

Now let $x \in N(D)$ and put $y = (I + (AG)^g AF)^{-1}(AG)^g AHx$. Then $y \in S$ and $y \in R((AG)^g) = AT$. So $y = 0$ and consequently, $(AG)^g AHx = 0$. But this means that $AHx \in AT \cap N((AG)^g) = AT \cap S = \{0\}$. Thus $AHx = 0$ and $Hx = 0$. Since $N(A) \cap T = \{0\}$, it follows that $x \in N(H) = S'$. Therefore,

$$A_{T,S}^{(2)} = A_{T,S}^{(2)}(I + (AG)^g AF)^{-1}(AG)^g AH.$$

Put $B' = I - AA_{T,S'}^{(2)}$, $B = I - AA_{T,S}^{(2)}$. Note that

$$W = A_{T,S}^{(2)} - A_{T,S'}^{(2)} = A_{T,S}^{(2)} - A_{T,S}^{(2)}AA_{T,S'}^{(2)} = A_{T,S}^{(2)}(AA_{T,S}^{(2)} - AA_{T,S'}^{(2)}).$$

So $W = A_{T,S}^{(2)}(AA_{T,S}^{(2)} - AA_{T,S'}^{(2)}) = A_{T,S}^{(2)}(B' - B)$. Since $B'\xi \in S'$, $\forall \xi \in Y$, we have $\text{dist}(B'\xi, S) \leq \delta(S', S)\|B'\xi\|$. Thus, for any $\epsilon > 0$, there is $u \in Y$ such that $\|B'\xi - Bu\| \leq \delta(S', S)\|B'\xi\| + \epsilon$ and so that

$$\|A_{T,S}^{(2)}(B'\xi - Bu)\| \leq \delta(S', S)\|B'\xi\|\|A_{T,S}^{(2)}\| + \|A_{T,S}^{(2)}\|\epsilon.$$

Noting that $A_{T,S}^{(2)}B = 0$, we have

$$\begin{aligned}
 \|W\xi\| &= \|A_{T,S}^{(2)}(B'\xi - B\xi)\| = \|A_{T,S}^{(2)}(B'\xi - Bu)\| \\
 &\leq \delta(S', S)\|B'\xi\|\|A_{T,S}^{(2)}\| + \|A_{T,S}^{(2)}\|\epsilon.
 \end{aligned}$$

But $\|B'\xi\| \leq \|\xi\| + \|A\|\|A_{T,S}^{(2)}\xi - W\xi\| \leq (1 + \kappa)\|\xi\| + \|A\|\|W\xi\|$. Thus,

$$(3.4) \quad \|W\xi\| \leq \delta(S', S)\|A_{T,S}^{(2)}\|((1 + \kappa)\|\xi\| + \|A\|\|W\xi\|) + \|A_{T,S}^{(2)}\|\epsilon.$$

(3.4) indicates that $\|W\| \leq \frac{(1 + \kappa)\hat{\delta}(S', S)}{1 - \kappa\hat{\delta}(S', S)}\|A_{T,S}^{(2)}\|$ and

$$\|A_{T,S'}^{(2)}\| \leq \|A_{T,S}^{(2)}\| + \|W\| \leq \frac{1 + \hat{\delta}(S', S)}{1 - \kappa\hat{\delta}(S', S)}\|A_{T,S}^{(2)}\|. \quad \square$$

We now present our main result of this paper as follows.

THEOREM 3.3. *Let $A \in B(X, Y)$ and let $T \subset X$, $S \subset Y$ be closed subspaces such that $A_{T,S}^{(2)}$ exists. Let $T' \subset X$, $S' \subset Y$ be closed subspaces such that $\hat{\delta}(T, T') < \frac{1}{(1 + \kappa)^2}$ and $\hat{\delta}(S, S') < \frac{1}{3 + \kappa}$ respectively. Then $A_{T',S'}^{(2)}$ exists and*

$$\begin{aligned} (1) \quad & A_{T',S'}^{(2)} = A_{T,S}^{(2)} + (I - A_{T,S}^{(2)}A)F(I + (AG)^gAF)^{-1}(AG)^g \\ & \quad + \{A_{T,S}^{(2)} + (I - A_{T,S}^{(2)}A)F(I + (AG)^gAF)^{-1}(AG)^g\} \\ & \quad \times (I + (A\tilde{G})^gA\tilde{F})^{-1}(A\tilde{G})^gA\tilde{F}(I - AA_{T,S}^{(2)})(I + AF(AG)^g)^{-1}. \\ (2) \quad & \|A_{T',S'}^{(2)} - A_{T,S}^{(2)}\| \leq \frac{(1 + \kappa)(\hat{\delta}(T, T') + \hat{\delta}(S', S))}{1 - (1 + \kappa)\hat{\delta}(T, T') - \kappa\hat{\delta}(S', S)}\|A_{T,S}^{(2)}\|. \\ (3) \quad & \|A_{T',S'}^{(2)}\| \leq \frac{1 + \hat{\delta}(S', S)}{1 - (1 + \kappa)\hat{\delta}(T, T') - \kappa\hat{\delta}(S', S)}\|A_{T,S}^{(2)}\|. \end{aligned}$$

where $G, \tilde{G}, \tilde{H} \in B(Y, X)$ are such that $R(G) = T$, $R(\tilde{G}) = R(\tilde{H}) = T'$, $N(G) = N(\tilde{G}) = S$, $N(\tilde{H}) = S'$ and $F = \tilde{G} - G$, $\tilde{F} = \tilde{H} - \tilde{G}$.

Proof. Since $\hat{\delta}(T, T') < \frac{1}{(1 + \kappa)^2}$, it follows from Lemma 3.1 that $A_{T',S}^{(2)}$ exists and

$$\begin{aligned} \|A\|\|A_{T',S}^{(2)}\| & \leq \frac{\kappa}{1 - (1 + \kappa)\hat{\delta}(T, T')} < 1 + \kappa \\ \hat{\delta}(S, S') & < \frac{1}{2 + 1 + \kappa} < \frac{1}{2 + \|A\|\|A_{T',S}^{(2)}\|}. \end{aligned}$$

Thus, by Lemma 3.2 we have that $A_{T',S'}^{(2)}$ exists and

$$(3.5) \quad A_{T',S'}^{(2)} = A_{T',S}^{(2)} + A_{T',S}^{(2)}(I + (A\tilde{G})_gA\tilde{F})^{-1}(A\tilde{G})_gA\tilde{F}(I - AA_{T',S}^{(2)})$$

by Lemma 3.2, where $\tilde{G}, \tilde{H} \in B(Y, X)$ with $R(\tilde{G}) = T'$, $N(\tilde{G}) = S$ and $R(\tilde{H}) = T'$, $N(\tilde{H}) = S'$ and $\tilde{F} = \tilde{H} - \tilde{G}$. By Lemma 3.1, we have

$$(3.6) \quad A_{T',S}^{(2)} = A_{T,S}^{(2)} + (I - A_{T,S}^{(2)}A)F(I + (AG)^gAF)^{-1}(AG)^gAA_{T,S}^{(2)}.$$

Combining (3.5) with (3.6), we get that

$$\begin{aligned} A_{T',S'}^{(2)} &= A_{T,S}^{(2)} + (I - A_{T,S}^{(2)}A)F(I + (AG)^g AF)^{-1}(AG)^g \\ &\quad + \{A_{T,S}^{(2)} + (I - A_{T,S}^{(2)}A)F(I + (AG)^g AF)^{-1}(AG)^g\} \\ &\quad \times (I + (A\tilde{G})^g A\tilde{F})^{-1}(A\tilde{G})^g A\tilde{F}(I - AA_{T,S}^{(2)})(I + AF(AG)^g)^{-1}. \end{aligned}$$

By Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned} \|A_{T',S'}^{(2)} - A_{T,S}^{(2)}\| &\leq \|A_{T',S'}^{(2)} - A_{T',S}^{(2)}\| + \|A_{T',S}^{(2)} - A_{T,S}^{(2)}\| \\ &\leq \frac{(1 + \|A\| \|A_{T',S}^{(2)}\|) \hat{\delta}(S', S)}{1 - \|A\| \|A_{T',S}^{(2)}\| \hat{\delta}(S', S)} \|A_{T',S}^{(2)}\| + \frac{(1 + \kappa) \hat{\delta}(T, T')}{1 - (1 + \kappa) \hat{\delta}(T, T')} \|A_{T,S}^{(2)}\| \\ &\leq \left[\frac{(1 + \kappa) \hat{\delta}(S', S) (1 - \hat{\delta}(T, T'))}{\{1 - (1 + \kappa) \hat{\delta}(T, T') - \kappa \hat{\delta}(S', S)\} \{1 - (1 + \kappa) \hat{\delta}(T, T')\}} \right. \\ &\quad \left. + \frac{(1 + \kappa) \hat{\delta}(T, T')}{1 - (1 + \kappa) \hat{\delta}(T, T')} \right] \|A_{T,S}^{(2)}\| \\ &= \frac{(1 + \kappa) (\hat{\delta}(T, T') + \hat{\delta}(S', S))}{1 - (1 + \kappa) \hat{\delta}(T, T') - \kappa \hat{\delta}(S', S)} \|A_{T,S}^{(2)}\|, \\ \|A_{T',S'}^{(2)}\| &\leq \|A_{T',S'}^{(2)} - A_{T,S}^{(2)}\| + \|A_{T,S}^{(2)}\| \\ &\leq \frac{(1 + \kappa) (\hat{\delta}(T, T') + \hat{\delta}(S', S))}{1 - (1 + \kappa) \hat{\delta}(T, T') - \kappa \hat{\delta}(S', S)} \|A_{T,S}^{(2)}\| + \|A_{T,S}^{(2)}\| \\ &\leq \frac{1 + \hat{\delta}(S', S)}{1 - (1 + \kappa) \hat{\delta}(T, T') - \kappa \hat{\delta}(S', S)} \|A_{T,S}^{(2)}\|. \quad \square \end{aligned}$$

LEMMA 3.4. *Let $A, \bar{A} = A + E \in B(X, Y)$ and $T \subset X, S \subset Y$ be closed subspaces such that $A_{T,S}^{(2)}$ exists. If $\|A_{T,S}^{(2)}\| \|E\| < 1$, then*

$$\bar{A}_{T,S}^{(2)} = (I + A_{T,S}^{(2)}E)^{-1} A_{T,S}^{(2)} = A_{T,S}^{(2)} (I + EA_{T,S}^{(2)})^{-1}.$$

and

$$\|\bar{A}_{T,S}^{(2)}\| \leq \frac{\|A_{T,S}^{(2)}\|}{1 - \|A_{T,S}^{(2)}\| \|E\|}, \quad \|\bar{A}_{T,S}^{(2)} - A_{T,S}^{(2)}\| \leq \frac{\|A_{T,S}^{(2)}\|^2 \|E\|}{1 - \|A_{T,S}^{(2)}\| \|E\|}.$$

Proof. $\|A_{T,S}^{(2)}\| \|E\| < 1$ implies that $(I + A_{T,S}^{(2)}E)^{-1}$ exists. Since

$$(I + A_{T,S}^{(2)}E)A_{T,S}^{(2)} = A_{T,S}^{(2)}(I + EA_{T,S}^{(2)}),$$

we have

$$(I + A_{T,S}^{(2)}E)^{-1} A_{T,S}^{(2)} = A_{T,S}^{(2)} (I + EA_{T,S}^{(2)})^{-1}.$$

Put $B = (1 + A_{T,S}^{(2)}E)^{-1}A_{T,S}^{(2)}$. Then $R(B) = R(A_{T,S}^{(2)}) = T$, $N(B) = N(A_{T,S}^{(2)}) = S$ and $B(A + E)B = B$. Therefore, $\bar{A}_{T,S}^{(2)} = (I + A_{T,S}^{(2)}E)^{-1}A_{T,S}^{(2)}$ with

$$\begin{aligned} \|\bar{A}_{T,S}^{(2)}\| &\leq \|(I + A_{T,S}^{(2)}E)^{-1}\| \|A_{T,S}^{(2)}\| \leq \frac{\|A_{T,S}^{(2)}\|}{1 - \|A_{T,S}^{(2)}\| \|E\|} \quad \text{and} \\ \|\bar{A}_{T,S}^{(2)} - A_{T,S}^{(2)}\| &= \|(I + A_{T,S}^{(2)}E)^{-1}A_{T,S}^{(2)}EA_{T,S}^{(2)}\| \leq \frac{\|A_{T,S}^{(2)}\|^2 \|E\|}{1 - \|A_{T,S}^{(2)}\| \|E\|}. \quad \square \end{aligned}$$

We close this section by giving the perturbation analysis for $A_{T,S}^{(2)}$ when T , S and A all have small perturbations.

THEOREM 3.5. *Let $A, \bar{A} = A + E \in B(X, Y)$ and let $T \subset X$, $S \subset Y$ be closed subspaces such that $A_{T,S}^{(2)}$ exists. Let $T' \subset X$, $S' \subset Y$ be closed subspaces with $\hat{\delta}(T, T') < \frac{1}{(1 + \kappa)^2}$ and $\hat{\delta}(S, S') < \frac{1}{3 + \kappa}$. Suppose that $\|A_{T,S}^{(2)}\| \|E\| < \frac{2\kappa}{(1 + \kappa)(4 + \kappa)}$. Then*

$$\begin{aligned} (1) \quad \bar{A}_{T',S'}^{(2)} &= [1 + A_{T,S}^{(2)}E + (I - A_{T,S}^{(2)}A)F(I + (AG)^gAF)^{-1}(AG)^gE \\ &\quad + \{A_{T,S}^{(2)} + (I - A_{T,S}^{(2)}A)F(I + (AG)^gAF)^{-1}(AG)^g\} \\ &\quad \times (I + (A\tilde{G})^gA\tilde{F})^{-1}(A\tilde{G})^gA\tilde{F}(I - AA_{T,S}^{(2)})(I + AF(AG)^g)^{-1}E]^{-1} \\ &\quad \times [A_{T,S}^{(2)} + (I - A_{T,S}^{(2)}A)F(I + (AG)^gAF)^{-1}(AG)^g \\ &\quad + \{A_{T,S}^{(2)} + (I - A_{T,S}^{(2)}A)F(I + (AG)^gAF)^{-1}(AG)^g\} \\ &\quad \times (I + (A\tilde{G})^gA\tilde{F})^{-1}(A\tilde{G})^gA\tilde{F}(I - AA_{T,S}^{(2)})(I + AF(AG)^g)^{-1}]. \\ (2) \quad \|\bar{A}_{T',S'}^{(2)}\| &\leq \frac{(1 + \hat{\delta}(S', S))\|A_{T,S}^{(2)}\|}{1 - (1 + \kappa)\hat{\delta}(T, T') - \kappa\hat{\delta}(S', S) - (1 + \hat{\delta}(S', S))\|A_{T,S}^{(2)}\| \|E\|}. \\ (3) \quad \frac{\|\bar{A}_{T',S'}^{(2)} - A_{T,S}^{(2)}\|}{\|A_{T,S}^{(2)}\|} &\leq \frac{(1 + \kappa)(\hat{\delta}(T, T') + \hat{\delta}(S', S) + (1 + \hat{\delta}(S', S))\|A_{T,S}^{(2)}\| \|E\|)}{1 - (1 + \kappa)\hat{\delta}(T, T') - \kappa\hat{\delta}(S', S) - (1 + \hat{\delta}(S', S))\|A_{T,S}^{(2)}\| \|E\|}. \end{aligned}$$

where, $F = \tilde{G} - G$, $\tilde{F} = \tilde{H} - \tilde{G}$ and $G, \tilde{G}, \tilde{H} \in B(Y, X)$ are arbitrary such that $R(G) = T$, $R(\tilde{G}) = R(\tilde{H}) = T'$, $N(G) = N(\tilde{G}) = S$ and $N(\tilde{H}) = S'$.

Proof. We have $A_{T',S'}^{(2)}$ exists and $\|A_{T',S'}^{(2)}\| \leq \frac{(1 + \hat{\delta}(S', S))\|A_{T,S}^{(2)}\|}{1 - (1 + \kappa)\hat{\delta}(T, T') - \kappa\hat{\delta}(S', S)}$ by Theorem 3.3. Thus, $\|A_{T',S'}^{(2)}\| \|E\| < \frac{(1 + \kappa)(4 + \kappa)\|A_{T,S}^{(2)}\| \|E\|}{2\kappa} < 1$ and hence $\bar{A}_{T',S'}^{(2)}$ exists with $\bar{A}_{T',S'}^{(2)} = (I + A_{T',S'}^{(2)}E)^{-1}A_{T',S'}^{(2)}$ by Lemma 3.4. It follows from Lemma

3.1 and 3.2 that

$$\begin{aligned}
\bar{A}_{T',S'}^{(2)} &= [I + A_{T,S}^{(2)}E + (I - A_{T,S}^{(2)}A)F(I + (AG)^g AF)^{-1}(AG)^g E \\
&\quad + \{A_{T,S}^{(2)} + (I - A_{T,S}^{(2)}A)F(I + (AG)^g AF)^{-1}(AG)^g\} \\
&\quad \times (I + (A\tilde{G})^g A\tilde{F})^{-1}(A\tilde{G})^g A\tilde{F}(I - AA_{T,S}^{(2)})(I + AF(AG)^g)^{-1}E]^{-1} \\
&\quad \times [A_{T,S}^{(2)} + (I - A_{T,S}^{(2)}A)F(I + (AG)^g AF)^{-1}(AG)^g \\
&\quad + \{A_{T,S}^{(2)} + (I - A_{T,S}^{(2)}A)F(I + (AG)^g AF)^{-1}(AG)^g\} \\
&\quad \times (I + (A\tilde{G})^g A\tilde{F})^{-1}(A\tilde{G})^g A\tilde{F}(I - AA_{T,S}^{(2)})(I + AF(AG)^g)^{-1}].
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\|\bar{A}_{T',S'}^{(2)}\| &\leq \frac{\|A_{T',S'}^{(2)}\|}{1 - \|A_{T',S'}^{(2)}\|\|E\|} \\
&\leq \frac{(1 + \hat{\delta}(S', S))\|A_{T,S}^{(2)}\|}{1 - (1 + \kappa)\hat{\delta}(T, T') - \kappa\hat{\delta}(S', S) - (1 + \hat{\delta}(S', S))\|A_{T,S}^{(2)}\|\|E\|}.
\end{aligned}$$

Notice that

$$\begin{aligned}
\bar{A}_{T',S'}^{(2)} - A_{T,S}^{(2)} &= (I + A_{T',S'}^{(2)}E)^{-1}A_{T',S'}^{(2)} - A_{T,S}^{(2)} \\
&= (I + A_{T',S'}^{(2)}E)^{-1}(A_{T',S'}^{(2)} - (I + A_{T',S'}^{(2)}E)A_{T,S}^{(2)}) \\
&= (I + A_{T',S'}^{(2)}E)^{-1}(A_{T',S'}^{(2)} - A_{T,S}^{(2)} - A_{T',S'}^{(2)}EA_{T,S}^{(2)}).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\|\bar{A}_{T',S'}^{(2)} - A_{T,S}^{(2)}\| &\leq \|(I + A_{T',S'}^{(2)}E)^{-1}\|(\|A_{T',S'}^{(2)} - A_{T,S}^{(2)}\| + \|A_{T',S'}^{(2)}EA_{T,S}^{(2)}\|) \\
&\leq \frac{1}{1 - \|A_{T',S'}^{(2)}\|\|E\|}(\|A_{T',S'}^{(2)} - A_{T,S}^{(2)}\| + \|A_{T',S'}^{(2)}\|\|E\|\|A_{T,S}^{(2)}\|) \\
&\leq \frac{(1 + \kappa)(\hat{\delta}(T, T') + \hat{\delta}(S', S) + (1 + \hat{\delta}(S', S))\|A_{T,S}^{(2)}\|\|E\|)}{1 - (1 + \kappa)\hat{\delta}(T, T') - \kappa\hat{\delta}(S', S) - (1 + \hat{\delta}(S', S))\|A_{T,S}^{(2)}\|\|E\|}\|A_{T,S}^{(2)}\|. \quad \square
\end{aligned}$$

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